in which

$$\Delta^2 = \frac{(n^2-1)^4}{\beta^4(n^2+1)^2} - \frac{2(n^2-1)^2}{\beta^2(n^2+1)}(n^2-1/\alpha^2\beta^2) + (n^2+1/\alpha^2\beta^2)^2 2\omega_{1,2}^2 = \frac{(n^2-1)^2}{\beta^2(n^2+1)} + n^2 + 1/\alpha^2\beta^2 \mp \Delta$$

$$\tau = ct/a,\ c^2 = E/\rho,\ \alpha^2 = h^2/12a^2,\ \beta^2 = E/kG$$

where E, G, and ρ are Young's modulus, shear modulus, and density; h and a are shell thickness and mean radius; t is time; k is the shear deflection coefficient taken as $k=5/6$; g is the acceleration due to gravity; and I is the peak intensity of impulse cosine distributed over $|\theta| < 90°$.

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Cylindrical Electrostatic Probes at Angles of Incidence

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Nomenclature

e	= charge on electron
k	= Boltzman constant
n^i	= ion number density, cm^{-3}
p	= pressure, torr
r_p	= probe radius, cm
T	= temperature
T^e	= electron temperature
V_p	= probe potential, volts
λ_D	= Debye length, cm
$\lambda_{\alpha\beta}$	= mean free path for species α and β

Subscripts

∞	= refers to property of freestream
0	= refers to stagnation value

Introduction

In the course of an investigation of conical electrostatic probes[1] it was important to give detailed examination to the current-voltage characteristics of cylindrical probes aligned with, and at angles of incidence to weakly ionized flows, since cylindrical probes were used to measure the charged-particle properties of the flows. The interest in cylindrical probe response stems from the fact that it has been found that the ion current collected by cylindrical probes sometimes exhibits a peak in the neighborhood of zero incidence, apparently for two quite different reasons. One cause of a current peak is the end effect identified by Bettinger and Chen[2] and Hester and Sonin,[3] which is associated with a nonnegligible contribution to the probe current from the end of the sheath facing the flow and is characterized by large values of λ_D/r_p. Another explanation which has been suggested for a current peak is that it is due to ion-ion collisions, which inhibit curved trajectory motions of the ions and cause them to move toward the probe in a more-or-less radial fashion. This situation would, of course, be obtained in quite a different range of probe operation than that for which the end effect was significant. Since it was possible to operate the cylindrical probe over a wide range of values of λ_{ii}/r_p, an examination of the effect of ion-ion collisions on probe response was deemed worthwile.

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Experimental Equipment

After ionization in a radio frequency induction heated torch, the test gas (argon) expanded through a converging-diverging nozzle with throat diameter 2.54 cm. The 1.22 m diam test section was maintained at low pressure by three stages of mechanical pumping and two oil booster pumps. Deionization of the flow by surface recombination was employed to vary the ion density by several orders of magnitude. Tantalum honeycombs 0.5 cm long were placed in a chamber which was added to the flow path between the torch exit and the nozzle. Electrons and ions in the flow diffuse to the metal surfaces of the honeycombs where the recombination occurs. The ion density of the flow could be lowered from about 10^{13} to less than 10^9 cm^{-3} by adding up to six of the honeycombs.

A device for varying the angle of incidence of a cylindrical electrostatic probe was built. Mounted on a traverse mechanism, it allowed the probe to be rotated from outside the chamber in such a fashion that the midpoint of the probe collecting surface was the center of rotation. An indicator on the device displayed the angle of incidence to the nearest 0.9° over a range from -90° to +90°. The cylindrical electrostatic probe was made from 0.038 cm diam tungsten wire with an exposed surface length of 0.749 cm. A 45 v battery pack supplied the probe voltage. Two electrometers connected to an x-y recorder measured the probe current-voltage characteristics.

Flow Calibration

The flow region used for the angle of incidence measurements was the exit of the diverging nozzle. The neutral flow properties at that point were determined from stagnation and impact pressure measurements together with stagnation temperature measurements by assuming an ideal gas with isentropic expansion through the nozzle and using the tables of Mueller.[4] The seven different flow conditions, characterized by the number of tantalum honeycomb sections (0-6) placed in the deionization chamber, are listed in Table 1. The electron temperatures for the two highest ion density conditions were measured with a double cylindrical probe using the method of Johnson and Malter.[5]

While the effects of ion-neutral collisions were negligible on the electron temperature measurements, they were important in the single probe measurements of ion densities. To account for these effects we used the mixing formula of Thornton[6] to calculate the ion densities from the measured single probe ion currents. For lower values of ion density, the theory of Laframboise[7] as presented by Sonin[8] was used to determine the collisionless current required by the Thornton formula. But at the highest ion density (no honeycomb) the cold ion theory of Allen, Boyd, and Reynolds[9] as applied to the cylinder by Chen[10] was used to calculate the collisionless current. We now believe that although λ_{in}/r_p may be large, when $(\lambda_{ii}/r_p) \ll 1$, the aligned probe current increases from that predicted by Laframboise to that predicted by Chen. We

Table 1 Flow conditions[a] for rotated cylindrical probe ($r_p = 0.019$ cm)

Number of honeycombs	T_0(K)	T_∞(K)	T^e_∞ (K)	M_∞	n^i_∞ (cm^{-3})	λ_D(cm)	λ_{nn}(cm)	λ_{in}(cm)	λ_{ii}(cm)
0	1480	582	1380	2.15	7.6×10^{12}	0.000096	0.049	0.022	0.0015
1	1030	405	1100	2.15	6.2×10^{9}	0.0029	0.031	0.015	0.58
2	1080	425	650	2.15	2.1×10^{9}	0.0039	0.030	0.016	1.9
3	900	352	430	2.16	1.3×10^{9}	0.0039	0.027	0.013	2.1
4	820	319	370	2.17	1.2×10^{9}	0.0051	0.024	0.012	1.8
5	730	284	300	2.17	9.0×10^{8}	0.0040	0.020	0.010	1.9
6	675	263	270	2.17	3.7×10^{8}	0.0059	0.018	0.0096	3.8

[a] $p_0 = 2.62$ torr for all the above. The values of the mean free paths were calculated according to formulas presented by Tseng.[12]

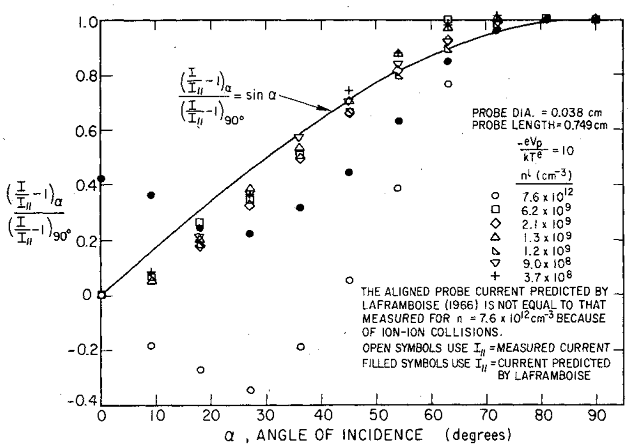

Fig. 1 Cylindrical probe current for different angles of incidence.

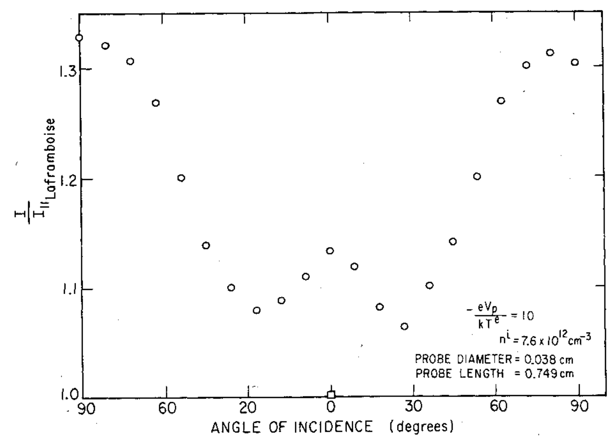

Fig. 2 Cylindrical probe current for different angles of incidence, $n^i = 7.6\times10^{12}$ cm.$^{-3}$

will show in the results section below that angle of incidence data are consistent with this increase in current.

The direct measurement of electron temperatures for the flows with 2-6 honeycombs was not possible because the double probe ion current had no well defined saturation and the logarithm of the single probe electron retarding field current plotted against probe voltage did not yield a straight line. Under these conditions, an approximate calculation of the electron temperature based on an electron energy balance was used to interpret the ion current measurements in terms of the ion densities of the flows. A more thorough description of this method appears in Appendix B of Ref. 11.

Results and Discussion

The current to the 0.038-cm diam cylindrical probe was measured for angles of incidence from 0-90°. For $n^i \leq 6.2\times10^9$ and $-eV_p/kT^e = 10$, the variation of the current to the probe for different angles was sinusoidal (see Fig. 1). At the highest ion density the variation was strikingly different, in that there appeared a peak ion probe current near zero angle of incidence.

This peak is not caused by the end effect because the data from flows with larger Debye lengths do not display such a peak. Ion-neutral collisions are not expected to be responsible because their mean free paths were moderately smaller for the lower ion density flows where no peak was observed. The most dramatic change in mean free path was that for the ion-ion collisions. For the highest ion density flows, where the peak was observed, $(\lambda_{ii}/r_p) \ll 1$, but for the other flows $(\lambda_{ii}/r_p) \gg 1$.

The ion-ion collisions are therefore believed to be responsible for the current peak. For this flow condition, therefore, the ion density was calculated from the aligned cylinder probe current using the cold ion theory of Chen with the correction for ion-neutral collisions from the Thornton formula. Then using the thus-determined ion density, the collisionless probe current was calculated from the theory of Laframboise, again using the Thornton formula to correct for ion-neutral collisions. This aligned probe current $I_{\text{Laframboise}}$ was used for the nondimensionalization in Fig. 2 and is plotted as the square symbol at $\alpha = 0$. It can be seen in Fig. 2 that if the current peak were not present, an extrapolation of the angle of incidence data to zero incidence would yield a current for the aligned probe close to the Laframboise value. Thus, the present measurements agree with the Hester and Sonin[3] explanation of Sonin's[8] earlier experiments; that is, for aligned cylindrical probes with $\lambda_{ii} \ll r_p$, the theory of Chen appears to properly account for ion-ion collisions.

Conclusions

The variation of cylindrical probe current for different angles of incidence was measured for seven flows with a large range of ion densities. For the highest ion density flow, where $\lambda_{ii} \ll r_p$, a current peak at zero incidence angle was observed. For these conditions the aligned probe current seems to be properly described by the cold ion theory of Chen.

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Quasi-Analytical Transient Conduction Solution

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Nomenclature

$[A]$	= matrix coefficient of finite difference or element model
A_{ij}	= elements of $[A]$
$[D]$	= diagonal matrix consisting of eigenvalues of $[A]$
F	= column matrix representing internal heat generation
L_i	= dimensions of laminated slab used in numerical example
M	= half bandwidth of $[A]$
N	= size of $[A]$
R_m	= remainder formula
$[S]$	= matrix composed of eigenvectors of $[A]$
T^*	= amplitude of prescribed surface temperature, Eq. (22)
T	= column matrix representing nodal temperatures
T_i	= temperature of ith node
T_0	= column matrix representing initial nodal temperatures
x_i	= Cartesian coordinates of laminated slab (see Fig. 1.)
ζ	= complex spatial portion of temperature field
θ_{ij}	= fiber orientations
$\kappa_{ij}/(\rho c_p)$	= thermal diffusivity (ft^2/sec)
Ω	= period of prescribed surface temperature, Eq. (22)
$Re\{\ \}$	= real part of $\{\ \}$
$\vert\ \vert$	= absolute value
$\Vert\ \Vert$	= norm ($\Vert [A] \Vert = \vert A_{11}\vert + \vert A_{12}\vert + \ldots + \vert A_{NN}\vert$)

Introduction

ALTHOUGH a great number of transient conduction problems have been solved, only a limited number of geometrical shapes and only those boundary conditions which can easily be expressed mathematically (Dirichlet) have been handled. Hence, for most transient conduction problems, one must resort to a numerical technique. Generally finite element and difference procedures are most prominently used. For such numerical models, the transient problem has been handled using numerical integration (predictor-corrector), Euler (forward), Crank-Nicolson (mid-difference), Galerkin algorithms, finite element,[1] and least squares stepping schemes[2] etc. As many of these procedures are conditionally or unconditionally stable, the choice of different time steps will lead to a variety of solution degradations. Recently, as an alternative approach, Dodd and Gupta[3] and Bahar[4] have used the state space procedures of atuomatic controls to obtain numerical solutions for several boundary value problems. Such procedures stem from a direct utilization of formal procedures of classical ODE theory. Based on one such formal solution, the present Note will develop a transient numerical procedure which will not be subject to the numerical degradation of the previously noted time stepping schemes. In fact, akin to Monte Carlo techniques,[5] the given procedure has the added capability of handling the temperature at a given node and/or nodes. As will be seen, this is possible without performing simultaneous calculations at all the nodes of the given model as is required of time stepping schemes.

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Equations and Solution

As a starting point, consider the following system of first-order ODE

$$dT/dt = [A]\,T + F(t);\ T(0) = T_0 \tag{1}$$

such that $[A]$ and $F(t)$ stem from either finite element or difference models wherein T represents the nodal temperatures, and T_0 is the initial condition column matrix. Following Ref. 6, assuming that $[A]$ is nonsingular, the formal exact solution of Eq. (1) is given by

$$T(t) = e^{t[A]}T_0 + \int_0^t e^{(t-\mu)[A]}F(\mu)\,d\mu \tag{2}$$

For the case in which the eigenvalues of $[A]$ are known, $e^{t[A]}$ can be reduced to the following canonical form,[6] namely

$$e^{t[A]} = [S]e^{t[D]}\,[S]^{-1} \tag{3}$$

As $[A]$ is in general a large matrix, the accurate evaluation of all of its eigenvalues and associated eigenvectors usually represents a formidable numerical problem. Hence, most investigators have resorted to a variety of time stepping schemes as noted earlier. For this Note, the numerical evaluation of Eq. (2) is obtained by formally expanding $e^{t[A]}$ in Maclaurin series, namely

$$e^{t[A]} = \sum_{n=0}^{\infty} (t^n/n!)\,[A]^n \tag{4}$$

Furthermore, to evaluate the convolution integral appearing in Eq. (2), since F is generally in the form of tabulated data, without loss in generality, the following polynomial spline fit is used for $t\epsilon[t_{i-1}, t_i]$, hence

$$F_i(t) \cong \sum_{\ell=0}^{L} F_{i\ell}t^{\ell} \tag{5}$$

such that

$$t_i = \sum_{j=0}^{i} \Delta t_j$$